Cambridge Assessment
Admissions Testing

STEP Mark Schemes 2017

Mathematics

STEP 9465/9470/9475

November 2017

Introduction

These mark schemes are published as an aid for teachers and students, and indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches; if a different approach was taken by a candidate, their solution was marked accordingly after discussion by the marking team. These adaptations are not recorded here.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

Admissions Testing will not enter into any discussion or correspondence in connection with this mark scheme.

Marking notation

NOTATION	MEANING	NOTES
M	Method mark	For correct application of a **M**ethod.
dM or m	Dependent method mark	This cannot be earned unless the preceding **M** mark has been earned.
A	Answer mark	**M0** ⇒ **A0**
B	Independently earned mark	Stand alone for “right or wrong”.
E	B mark for an explanation	
G	B mark for a graph	
ft	Follow through	To highlight where incorrect answers should be marked as if they were correct.
CAO or **CSO** **Sometimes written as A***	Correct Answer/Solution Only	To emphasise that **ft** does not apply.
AG	Answer Given	Indicates answer is given in question.

Question 1

(i) $I_n = \int_0^1 \arctan x \,.\, x^n \, dx$ **M1** Use of intgrn. by parts (parts correct way round)

$= \left[\arctan x \,.\, \frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2} . \frac{x^{n+1}}{n+1} \, dx$ **A1** Correct to here

$= \left(\frac{\pi}{4} . \frac{1}{n+1} - 0\right) - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x^2} \, dx$

$\Rightarrow (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} \, dx$ **A1** Given Answer legitimately established **3**

Setting $n = 0$, $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx$ **M1** Attempt to solve this using recognition/ substitution

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]$ **M1** Log integral involved

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ **A1 CAO** **3**

(ii) $n \to n+2$ in given result:

$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2} \, dx$ **B1** Noted or used somewhere

$(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2} \, dx$ **M1** Adding and cancelling ready to integrate

$= \frac{\pi}{2} - \frac{1}{n+2}$ **A1 CAO** **3**

Setting $n = 0$ and then $n = 2$ in this result (or equivalent involving integrals):

$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$ and $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$ **M1**

Eliminating I_2 and using value for I_0 to find I_4 **M1** By subtracting, or equivalent

$I_4 = \frac{1}{20}\left(1 + \pi - 2\ln 2\right)$ **A1 FT** from their I_0 value **3**

(iii) For $n = 1$, $5I_4 = A - \frac{1}{2}\left(-1 + \frac{1}{2}\right) = A + \frac{1}{4}$

$= \frac{1}{4} + \frac{1}{4}\pi - \frac{1}{2}\ln 2$ **M1** Comparing formula with found I_4 value

and the result is true for $n = 1$ provided

$A = \frac{1}{4}\pi - \frac{1}{2}\ln 2$ **A1 FT** from their I_4 value **2**

Assuming $(4k+1)I_{4k+1} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r}$ **M1** For a clearly stated induction hypothesis

(or a fully explained "if … then …" at end)

$(4k+5)I_{4k+4} + (4k+3)I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4}$ **B1**

$(4k+3)I_{4k+2} + (4k+1)I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2}$ **B1**

Subtracting:

$(4k+5)I_{4k+4} = (4k+1)I_{4k} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1**

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1** Use of assumed result

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} - \frac{1}{2}(-1)^{2k+1}\frac{1}{2k+1} - \frac{1}{2}(-1)^{2k+2}\frac{1}{2k+2}$

$= A - \frac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r \frac{1}{r}$ **A1** A clear demonstration of how the two extra terms fit must be given **6**

Question 2

Let $x_n = X$. Then $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$ **M1 A1** Correct, unsimplified

i.e. $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$ **M1** Attempt to remove "fractions within fractions"

A1 Correct, simplified **4**

(i) If $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX$ **M1**

$\Rightarrow\ 0 = X^2 - (a - b)X + 1$ **A1**

If $x_{n+2} = x_n$ then

$(a^2 - 1)X - (a + b) = (a + b)X^2 + (b^2 - 1)X$ **M1**

$\Rightarrow\ 0 = (a + b)\{X^2 - (a - b)X + 1\}$ **M1 A1** Factorisation

and so, for $x_{n+2} = x_n$ but $x_{n+1} \neq x_n$

we must have $a + b = 0$ **A1 Given Answer** fully justified & clearly stated

(No marks for setting $b = -a$, for instance, and showing sufficiency)

For "comparing coefficients" approach (must be all 3 terms) max. 3/4 **6**

(ii) $x_{n+4} = \dfrac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)}$ **M1** Use of the two-step result from earlier

$$= \frac{(a^2-1)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]-(a+b)}{(a+b)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]+(b^2-1)}$$

A1 Correct, unsimplified, in terms of X

If $x_{n+4} = x_n$ then **M1** Equating

$(a^2 - 1)^2X - (a + b)(a^2 - 1) - (a + b)^2X - (a + b)(b^2 - 1)$ **A1** LHS correct

$= (a + b)(a^2 - 1)X^2 - (a + b)^2X + (a + b)(b^2 - 1)X^2 + (b^2 - 1)^2X$ **A1** RHS correct

$\Rightarrow\ 0 = (a + b)(a^2 + b^2 - 2)X^2 - [(a^2 - 1)^2 - (b^2 - 1)^2]X + (a + b)(a^2 + b^2 - 2)$

M1 Good attempt to simplify

$\Rightarrow\ 0 = (a + b)(a^2 + b^2 - 2)\{X^2 - (a - b)X + 1\}$ **M1** Factorisation attempt

A1 A1 Partial; complete

and the sequence has period 4 if and only if

$a^2 + b^2 = 2,\ a + b \neq 0,\ X^2 - (a - b)X + 1 \neq 0$ **B1 CAO** Correct final statement

[Ignore any discussion or confusion regarding issues of necessity and sufficiency]

NB Some candidates may use the one-step result repeatedly and get to x_{n+4} via x_{n+3}:

$x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2++ab+b^2-1)X-(a+2b+b^3)}$ and $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$ starts the process; then as above.

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ALT. Consider the two-step sequence $\{\dots, x_n, x_{n+2}, x_{n+4}, \dots\}$ given by (assuming $a + b \neq 0$)

$$x_{n+2} = \frac{\left(\dfrac{a^2-1}{a+b}\right)X - 1}{X + \left(\dfrac{b^2-1}{a+b}\right)} \equiv \frac{AX-1}{X+B}$$, which is clearly of exactly the same form as before.

Then $x_{n+4} = x_n$ if and only if $a + b \neq 0$, $X^2 - (a-b)X + 1 \neq 0$ (from $x_{n+4} \neq x_{n+2}$ and $x_{n+4} \neq x_n$ as before), together with the condition $A + B = 0$ (also from previous work); i.e. $\dfrac{a^2-1}{a+b} + \dfrac{b^2-1}{a+b} = 0$, which is equivalent to $a^2 + b^2 - 2 = 0$ since $a + b \neq 0$.

Note that it is not necessary to consider $x_{n+4} \neq x_{n+3}$ since if $x_{n+4} = x_{n+3} = X$ then the sequence would be constant.

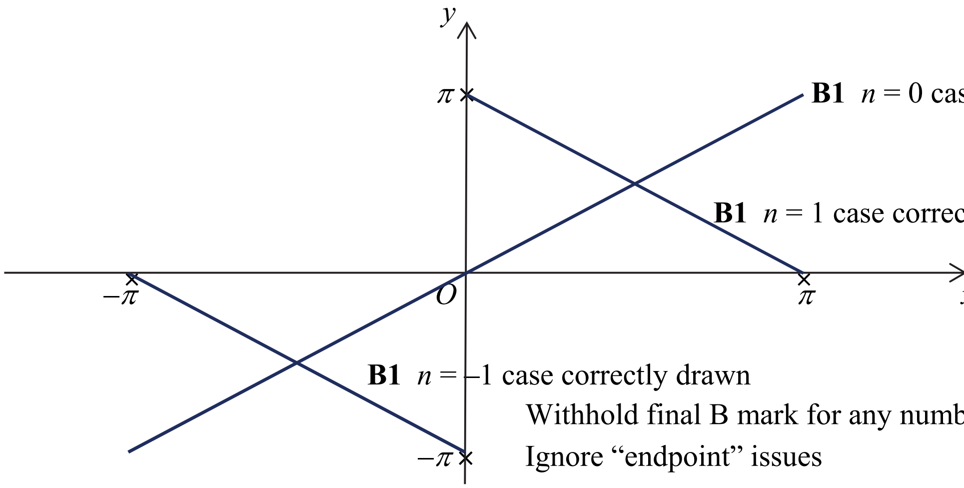
Question 3

(i) $\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x$

$n = -1$: $y = -\pi - x$ **B1**

$n = 0$: $y = x$ **B1**

$n = 1$: $y = \pi - x$ **B1** Withhold final B mark for any number of extra eqns.



B1 $n = 0$ case correctly drawn

B1 $n = 1$ case correctly drawn

B1 $n = -1$ case correctly drawn

Withhold final B mark for any number of extra lines

Ignore "endpoint" issues

6

(ii) $\sin y = \frac{1}{2}\sin x \Rightarrow \cos y \frac{dy}{dx} = \frac{1}{2}\cos x$ **M1** Implicit diffn. attempt (or equivalent)

$\frac{dy}{dx} = \frac{\cos x}{2\cos y}$ **A1** Correct

$= \frac{\cos x}{2\sqrt{1 - \frac{1}{4}\sin^2 x}}$ or $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$ **A1** Correct and in terms of x only

3

$$\frac{d^2 y}{dx^2} = \frac{(4 - \sin^2 x)^{\frac{1}{2}}.-\sin x - \cos x.\frac{1}{2}(4 - \sin^2 x)^{-\frac{1}{2}}.-2\sin x\cos x}{4 - \sin^2 x}$$

M1 For use of the *Quotient Rule* (or equivalent)

M1 For use of the *Chain Rule* for d/dx(denominator)

A1

$$= \frac{-\sin x(4 - \sin^2 x) + \cos^2 x.\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

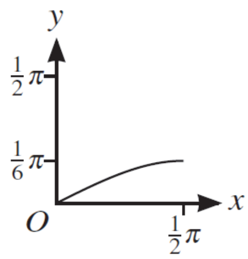
M1 Method for getting correct denominator

$$= \frac{\sin x\{\cos^2 x - 4 + \sin^2 x\}}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

$$= \frac{-3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

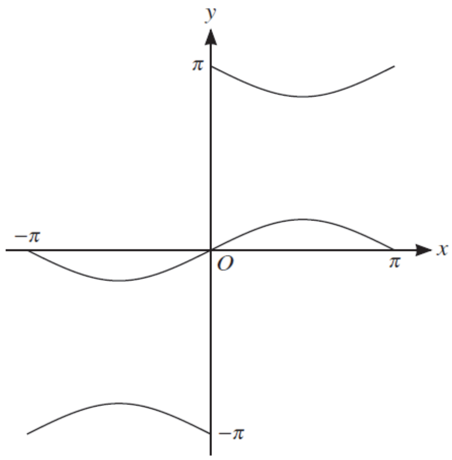
A1 **Given Answer** correctly obtained from $c^2 + s^2 = 1$

5



Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0) increasing to a maximum at $(\frac{\pi}{2}, \frac{\pi}{6})$ since $\frac{d^2y}{dx^2} < 0$

B1 (Gradient and coordinate details unimportant unless graphs look silly as a result)



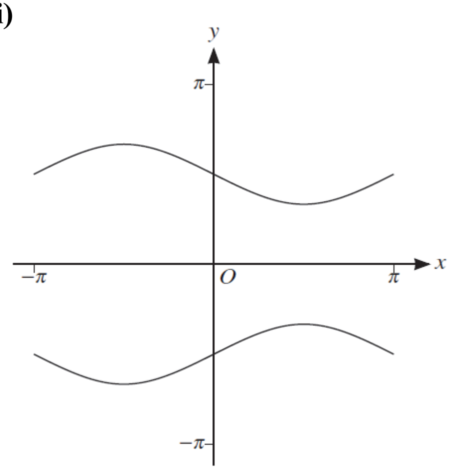
B1 Reflection symmetry in $x = \frac{\pi}{2}$

B1 Rotational symmetry about O

B1 Reflection symmetry in $y = \pm\frac{\pi}{2}$

4

(iii)



B1 RHS correct

B1 LHS correct

2

Question 4

(i) Setting f(x) = 1 in **(*)** gives

$$\left(\int_a^b g(x)\,dx\right)^2 \le \left(\int_a^b 1\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right)$$ **B1** Clearly stated

Let g(x) = e^x : $\left(\int_a^b e^x\,dx\right)^2 \le (b-a)\left(\int_a^b e^{2x}\,dx\right)$ **M1**

$\Rightarrow\ \left(e^b - e^a\right)^2 \le (b-a).\tfrac{1}{2}\left(e^{2b} - e^{2a}\right)$

$\Rightarrow\ \left(e^b - e^a\right)^2 \le (b-a).\tfrac{1}{2}\left(e^b - e^a\right)\left(e^b + e^a\right)$

$\Rightarrow\ e^b - e^a \le \tfrac{1}{2}(b-a)\left(e^b + e^a\right)$ **A1**

Choosing $a = 0$ and $b = t$ gives **M1**

$e^t - 1 \le \tfrac{1}{2}t\left(e^t + 1\right)\ \Rightarrow\ \dfrac{e^t - 1}{e^t + 1} \le \tfrac{1}{2}t$ **A1** **Given Answer** legitimately obtained **5**

(ii) Setting f(x) = x, a = 0 and b = 1 in **(*)** gives

$$\left(\int_0^1 x\,g(x)\,dx\right)^2 \le \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 [g(x)]^2\,dx\right)$$ **B1** Clearly stated

Choosing g(x) = $e^{-\frac{1}{4}x^2}$ gives **M1**

$$\left(\int_0^1 xe^{-\frac{1}{4}x^2}\,dx\right)^2 \le \tfrac{1}{3}\left(1^3 - 0^3\right)\left(\int_0^1 e^{-\frac{1}{2}x^2}\,dx\right)$$

$$\left(\left[-2e^{-\frac{1}{4}x^2}\right]_0^1\right)^2 \le \tfrac{1}{3}\left(\int_0^1 e^{-\frac{1}{2}x^2}\,dx\right)$$ **A1 A1** LHS, RHS correct

$$\Rightarrow\ \int_0^1 e^{-\frac{1}{2}x^2}\,dx \ge 3\left(-2\left[-e^{-\frac{1}{4}} + 1\right]\right)^2$$

i.e. $\displaystyle\int_0^1 e^{-\frac{1}{2}x^2}\,dx \ge 12\left(1 - e^{-\frac{1}{4}}\right)^2$ **A1** **Given Answer** legitimately obtained **5**

(iii) With f(x) = 1, g(x) = $\sqrt{\sin x}$, a = 0, b = $\tfrac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) becomes **M1** Any sensible f, g used in **(*)**

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right)^2 \le \tfrac{1}{2}\pi\left(\int_0^{\frac{1}{2}\pi} \sin x\,dx\right)$$ **A1**

RHS is $\tfrac{1}{2}\pi\left[-\cos x\right]_0^1 = \tfrac{1}{2}\pi$

(and since LHS is positive) we have $\displaystyle\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx \le \sqrt{\frac{\pi}{2}}$

A1 RH half of **Given** inequality obtained from fully correct working **4**

With $f(x) = \cos x$, $g(x) = \sqrt[4]{\sin x}$, $a = 0$, $b = \frac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) gives

$$\left(\int_0^{\frac{1}{2}\pi} \cos x.(\sin x)^{\frac{1}{4}}\, dx\right)^2 \le \left(\int_0^{\frac{1}{2}\pi} \cos^2 x\, dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx\right)$$ **A1**

$$\text{LHS} = \left(\left[\tfrac{4}{5}(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{16}{25}$$ **M1 A1** By recognition/substitution integration

and $\displaystyle\int_0^{\frac{1}{2}\pi} \cos^2 x\, dx = \int_0^{\frac{1}{2}\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$ **M1**

$$= \left(\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{1}{4}\pi$$ **A1**

Giving the required LH half of the **Given** inequality:

$$\tfrac{16}{25} \le \tfrac{1}{4}\pi\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx\right) \quad \text{i.e.} \quad \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx \ge \frac{64}{25\pi}$$

6

Withhold the last A mark if final result is not arrived at

Question 5

(i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2a}{2at} = \dfrac{1}{t}$ — **M1** Finding gradt. of tgt. (or by implicit diffn.)

$\Rightarrow$ Grad. nml. at P is $-p$ — **A1**

$\Rightarrow$ Eqn. nml. to C at P is $x - 2ap = -p(x - ap^2)$ — **B1 FT** any form, e.g. $y = -px + ap(2 + p^2)$

Nml. meets C again when $x = an^2$, $y = 2an$ — **M1** Substd. into nml. eqn.

$\Rightarrow$ $2an = -pan^2 + ap(2 + p^2)$

$\Rightarrow$ $0 = pn^2 + 2n - p(2 + p^2)$

$\Rightarrow$ $0 = (n - p)(pn + [2 + p^2])$ — **M1** Solving attempt

Since $n = p$ at P, it follows that $n = -\dfrac{2+p^2}{p}$ at N

i.e. $n = -\left(p + \dfrac{2}{p}\right)$ — **A1 Given Answer** legitimately obtained **6**

(ii) Distance $P(ap^2, 2ap)$ to $N(an^2, 2an)$ is given by

$PN^2 = \left[a(p^2 - n^2)\right]^2 + \left[2a(p-n)\right]^2$ — **M1**

$= a^2(p-n)^2\left\{(p+n)^2 + 4\right\}$

$= a^2\left(2p + \dfrac{2}{p}\right)^2\left\{\left(\dfrac{-2}{p}\right)^2 + 4\right\}$ — **M1** Substituting for n

$= 16a^2\left(\dfrac{p^2+1}{p}\right)^2\left\{\dfrac{1+p^2}{p}\right\} = 16a^2\dfrac{(p^2+1)^3}{p^4}$ — **A1 Given Answer** legitimately obtained **3**

$\dfrac{d(PN^2)}{dp} = 16a^2\dfrac{d(p^2 + 3 + 3p^{-2} + p^{-4})}{dp}$ — **M1** Differentiation directly, or by the Quotient Rule

$= 16a^2(2p - 6p^{-3} - 4p^{-5})$ — **A1** Correct, unsimplified

$= 32a^2\dfrac{p^6 - 3p^2 - 2}{p^5}$

$= \dfrac{32a^2}{p^5}(p^2+1)^2\left[p^2 - 2\right]$

Note that $\dfrac{d(PN^2)}{dp} = 16a^2\left\{\dfrac{p^4.3(p^2+1)^2.2p - (p^2+1)^3.4p^3}{p^8}\right\}$

$= \dfrac{32a^2}{p^8}.p^3(p^2+1)^2\left[3p^2 - 2(p^2+1)\right]$ by the Quotient Rule

$\dfrac{d(PN^2)}{dp} = 0$ only when $p^2 = 2$ — **A1 Given Answer** fully shown

Justification that it is a minimum — **E1**

(either by examining the sign of $\dfrac{d(PN^2)}{dp}$

or by explaining that PN^2 cannot be maximised **4**

(iii) Grad. PQ is $\frac{2}{p+q}$ **B1**

Grad. NQ is $\frac{2}{n+q}$ or $\frac{2}{q-p-\frac{2}{p}}$ **B1**

Since $\angle PQN = 90°$ (by "$\angle$ in a semi-circle"; i.e. *Thales Theorem*)

$$\frac{2}{p+q} \times \frac{2}{q-p-\frac{2}{p}} = -1$$ **M1**

$$\Rightarrow \ 4 = (p+q)\left(p-q+\frac{2}{p}\right) = p^2 - q^2 + 2 + \frac{2q}{p}$$

$$\Rightarrow \ 2 = p^2 - q^2 + \frac{2q}{p}$$ **A1 Given Answer** legitimately obtained **4**

PN minimised when $p^2 = 2 \ \Rightarrow \ q^2 = \frac{2q}{p}$ **M1** Substituted into given expression

$\Rightarrow \ q = 0$ or $q = \frac{2}{p} = \pm\sqrt{2}$ **A1**

But $q = \pm\sqrt{2} \ \Rightarrow q = p$ (which is not the case) **E1** Other cases must be ruled out

Special Case: 1/3 for substg. $q = 0$ and verifying that $p^2 = 2$ **3**

Question 6

(i)		
When $n = 1$ $$S_1 = 1 \leq 2\sqrt{1} - 1$$	**B1**	Clear verification.
Assume that the statement is true when $n = k$: $$S_k \leq 2\sqrt{k} - 1$$	**B1**	Must be clear that this is assumed.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$$	**M1**	Linking S_{k+1} and S_k
$$\leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$	**M1**	Using assumed result
Sufficient to prove: $$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$	**M1**	
i.e. $2\sqrt{k(k+1)} + 1 \leq 2(k+1)$	**A1**	Multiplying by $\sqrt{k+1}$ or putting over a common denominator
i.e. $2\sqrt{k(k+1)} \leq 2k + 1$		
i.e. $4k^2 + 4k \leq 4k^2 + 4k + 1$	**A1**	
Which is clearly true. Therefore by induction the statement is true for all $n \geq 1$.	**B1**	Clear conclusion showing logic of induction.
	[8]	
(ii)		
Required to prove: $$(4k+1)^2(k+1) > (4k+3)^2 k$$	**M2**	Squaring given inequality
i.e. $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$ which is clearly true.	**A1**	
	[3]	
When $n = 1$: $$S_1 = 1 \geq 2 + \frac{1}{2} - c$$	**M1**	
So we need $c \geq \frac{3}{2}$	**A1**	
Prove $c = \frac{3}{2}$ works using induction	**M1**	
Assume holds when $n = k$: $$S_k \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2}$$	**M1**	Allow a general c.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c$$	**M1**	
Sufficient to prove: $$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - c$$	**A1**	
i.e. $4k\sqrt{k+1} + \sqrt{k+1} + 2\sqrt{k} \geq 4\sqrt{k}(k+1) + \sqrt{k}$	**A1A1**	
Which simplifies to the previously proved inequality. No further restrictions on c, so the minimum value is $c = \frac{3}{2}$	**B1**	
	[9]	

Question 7

(i) For $0 < x < 1$, x is positive and $\ln x$ is negative
so $0 > x \ln x > \ln x$
$\Rightarrow$ $e^0 > e^{x \ln x} > e^{\ln x}$ or $\ln 1 > \ln x^x > \ln x$
$\Rightarrow$ $(1 >)\ f(x) > x$ since ln is a strictly increasing fn. **B1**
Again, since $\ln x < 0$, it follows that
$\ln x < f(x) \ln x < x \ln x$
$\Rightarrow$ $\ln x < \ln\{g(x)\} < \ln\{f(x)\}$ **M1** Suitably coherent justification
$\Rightarrow$ $x < g(x) < f(x)$ **A1** **Given Answer** legitimately obtained

For $x > 1$, $\ln x > 0$ and so $x < f(x) < g(x)$ **B1** No justification required **4**

(ii) $\ln\{f(x)\} = x \ln x$ **M1** Taking logs and attempting implicit diffn.
Alt. Writing $y = e^{x \ln x}$ and diffg.

$\frac{1}{f(x)}.f'(x) = x.\frac{1}{x} + 1.\ln x$ i.e. $f'(x) = (1 + \ln x)f(x)$ **A1**

$f'(x) = 0$ when $1 + \ln x = 0$, $\ln x = -1$, $x = e^{-1}$ **A1** **3**

(iii) $\lim_{x \to 0} (f(x)) = \lim_{x \to 0} (e^{x \ln x}) = \lim_{x \to 0} (e^0) = 1$ **B1** Suitably justified

$\lim_{x \to 0} (g(x)) = \lim_{x \to 0} (x^{f(x)}) = \lim_{x \to 0} (x^1) = 0$ **B1** May just be stated

Alt. $\lim_{x \to 0} (g(x)) = \lim_{x \to 0} (e^{f(x)\ln x}) = \lim_{x \to 0} (e^{\ln x}) = \lim_{x \to 0} (x) = 0$ **2**

(iv) For $y = \frac{1}{x} + \ln x \ (x > 0)$,

$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$ or $\frac{x-1}{x^2} = 0$ … **M1** Diffg. and equating to zero

… when $x = 1$ **A1** From correct derivative

For $x = 1-$, $\frac{dy}{dx} < 0$ and for $x = 1+$, $\frac{dy}{dx} > 0$ **M1** Method for deciding

(1, 1) is a MINIMUM of $y = \frac{1}{x} + \ln x$ **A1**

(Since there are no other TPs or discontinuities)
$y \geq 1$ for all $x > 0$ Conclusion must be made for all 4 marks **4**

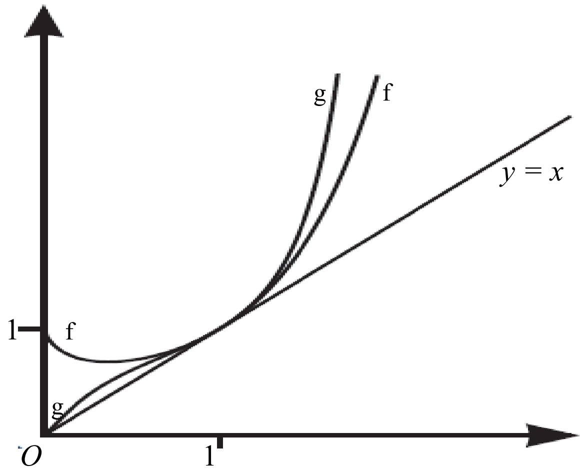
$\ln(g(x)) = f(x) \ln x$ **M1** Taking logs and attempting implicit diffn.

$\frac{1}{g(x)}.g'(x) = f(x).\frac{1}{x} + \ln x\{f(x)(1 + \ln x)\}$ **A1** using $f'(x)$ from **(ii)**

$\Rightarrow$ $g'(x) = f(x).g(x)\left\{\frac{1}{x} + \ln x + (\ln x)^2\right\}$

$\geq f(x).g(x)\{1 + (\ln x)^2\}$ **M1** using previous result of **(iv)**

> 0 since f, g > 0 from **(i)**
and $1 + (\ln x)^2 \geq 1 > 0$ **A1** **Given Answer** fully justified **4**



B1 One of f, g correct …
B1 Both correct …
… relative to $y = x$

B1 All three passing thro' (1, 1)

3

Question 8

Line thro' A perpr. to BC is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{u}$ — **B1**

Line thro' B perpr. to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$ — **B1**

Lines meet when $(\mathbf{r} = \mathbf{p} =)\ \mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v}$ — **M1** Equated

$\Rightarrow\ \mathbf{v} = \dfrac{1}{\mu}(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u})$ — **A1**

Since $\mathbf{v}$ is perpr. to CA, $(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u})\bullet(\mathbf{a} - \mathbf{c}) = 0$	**M1**

$\Rightarrow\ (\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{c}) + \lambda\,\mathbf{u}\bullet(\mathbf{a}-\mathbf{c}) = 0$ — **A1** Correctly multiplied out

$\Rightarrow\ \lambda = \dfrac{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{a}-\mathbf{c})}{\mathbf{u}\bullet(\mathbf{a}-\mathbf{c})}$ — **M1** Re-arranging for λ

A1 Correct (any sensible form)

$\Rightarrow\ \mathbf{p} = \mathbf{a} + \left(\dfrac{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{a}-\mathbf{c})}{\mathbf{u}\bullet(\mathbf{a}-\mathbf{c})}\right)\mathbf{u}$ — **A1** **FT** their λ (if only **a**, **b**, **c**, **u** involved)

9

$\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$ — **B1** **FT** their λ

Attempt at $\overrightarrow{CP}\bullet\overrightarrow{AB}$ — **M1**

$= (\mathbf{a}-\mathbf{c}+\lambda\mathbf{u})\bullet(\mathbf{b}-\mathbf{a})$ — **A1** Correct to here

$= (\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{b}-\mathbf{a})$

Now $\mathbf{u}\bullet(\mathbf{b}-\mathbf{c}) = 0$ since $\mathbf{u}$ perpr. to BC — **M1**

$\Rightarrow\ \mathbf{u}\bullet\mathbf{b} = \mathbf{u}\bullet\mathbf{c}$ — **A1**

so that $\overrightarrow{CP}\bullet\overrightarrow{AB} = (\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{c}-\mathbf{a})$ — **M1** Substituted in

$= (\mathbf{a}-\mathbf{c})\bullet(\mathbf{a}-\mathbf{b}+\lambda\mathbf{u})$ — **M1 A1** Factorisation attempt; correct

$= 0$ from boxed line above — **A1 E1** Statement; justified

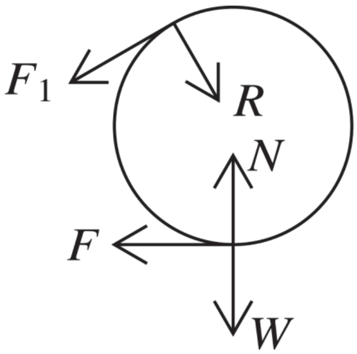
$\Rightarrow$ CP is perpr. to AB — **E1** For final, justified statement

11

Notice that the "value" of is never actually required

Any candidate who states the result is true because P is the *orthocentre* of ΔABC may be awarded **B2** for actually knowing something about triangle-geometry, but only in addition to any of the first 3 marks earned in the above solution: i.e. a maximum of 5/11 for the second part of the question.

Question 9

(i) 		
<u>$O\circlearrowleft$</u> $\quad F\,.\,r = F_1\,.\,r$ $\Rightarrow\ F = F_1$	**B1**	For correct moment equation.
<u>Res. $\leftrightarrow$</u> $F + F_1\cos\theta = R\sin\theta$	**B1**	For resolving horizontally for one cylinder.
Together give $R\sin\theta = F(1 + \cos\theta)$	**AG**	
Since $F_1 \le \mu R$, with $\mu = \frac{1}{2}$,	**M1**	Use of the Friction law
it follows that $\dfrac{F}{R} \le \frac{1}{2} \ \Rightarrow\ \dfrac{\sin\theta}{1+\cos\theta} \le \frac{1}{2}$	**A1**	Combining with previous answer
i.e. $2\sin\theta \le 1 + \cos\theta$	**AG** **Subtotal: 4**	
(ii) <u>Res. $\uparrow$ for RH cylinder</u> $W = N - R\cos\theta - F_1\sin\theta$	**B1**	F_1 might correctly be replaced with F.
<u>Res. $\uparrow$ for plank</u> $kW = 2R\cos\theta + 2F\sin\theta$	**B1**	
<u>Eliminating W:</u> $k(N - R\cos\theta - F\sin\theta) = 2R\cos\theta + 2F\sin\theta$	**M1**	For eliminating W
$N = R\cos\theta\left(\dfrac{2}{k}+1\right) + F\sin\theta\left(\dfrac{2}{k}+1\right)$	**A1**	For correct rearrangement for N
$N = \left(\dfrac{2}{k}+1\right)\left(\dfrac{1+\cos\theta}{\sin\theta}.\cos\theta + \sin\theta\right)F$	**M1**	For use of $R = \left(\dfrac{1+\cos\theta}{\sin\theta}\right)F$
$N = \left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta}\right)F$	**B1**	Obtaining $\cos^2\theta + \sin^2\theta$
$= \left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta+1}{\sin\theta}\right)F$	**AG**	
For no slipping at the ground, $F \le \mu N$	**M1**	Using Friction equation
$\Rightarrow\ F \le \frac{1}{2}\left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta+1}{\sin\theta}\right)F$	**A1**	Using previous part
ie. $2k\sin\theta \le (k+2)(1+\cos\theta)$	**A1**	Rearranging into a “useful” form.

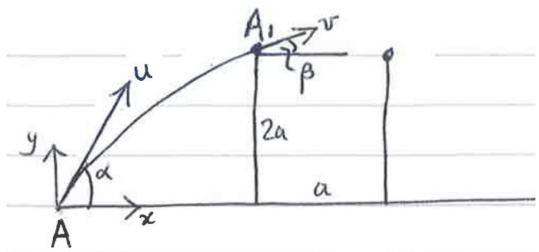
However, we already have that $2k\sin\theta \le k(1+\cos\theta) \le (k+2)(1+\cos\theta)$ so there are no extra restrictions on θ.	**E1** **Subtotal: 10**	Properly justified
(iii) $4\sin^2\theta \le 1 + 2\cos\theta + cos^2\theta$ $4(1-\cos^2\theta) \le 1 + 2\cos\theta + \cos^2\theta$ $0 \le 5\cos^2\theta + 2\cos\theta - 3$ $0 \le (5\cos\theta - 3)(\cos\theta + 1)$ Since $\cos\theta \ge 0$ we have $\cos\theta \ge \frac{3}{5}$ For appropriate angles $\cos\theta$ is decreasing and $\sin\theta$ is increasing. Therefore $\sin\theta \le \frac{4}{5}$ $\sin\theta = \dfrac{r-a}{r}$ So $5r - 5a \le 4r$ $r \le 5a$	**M1** **M1** **A1** **E1** **AG** **B1** **M1** **AG** **Subtotal: 6**	Squaring up an appropriate trig inequality Creating and simplifying quadratic inequality in one trig ratio A graphical argument is perfectly acceptable here. N.b It is possible that inequalities like $2s - 1 \le c$ are squared. If this is done without justifying that both sides are positive then withhold this final **E1**. Combining with previous result

Question 10

$ma = F - (Av^2 + R)$ $\text{WD} = \int_0^d F \, \mathrm{d}x$ $= \int_0^d \left(ma + Av^2 + R\right)\mathrm{d}x$ Since $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ $\text{WD} = \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\dfrac{\mathrm{d}x}{\mathrm{d}v}\,\mathrm{d}v$ $= \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\dfrac{v}{a}\,\mathrm{d}v$ Using $v^2 = u^2 + 2as$ with $v = w, u = 0, s = d \Rightarrow$ $w = \sqrt{2ad}$ Therefore: $\text{WD} = \int_{v=0}^{v=w} \dfrac{\left(ma + Av^2 + R\right)v}{a}\,\mathrm{d}v$	**B1** **M1** **AG** **B1** **M1** **B1** **AG** **[5]**	Clear use of N2L Attempting to change variable of integration. Justifying limits. Ignore absence of ±
(i) $\text{WD} = \left[\left(m + \dfrac{R}{a}\right)\dfrac{v^2}{2} + \dfrac{Av^4}{4a}\right]_0^{\sqrt{2ad}}$ $= \left(m + \dfrac{R}{a}\right)ad + Aad^2$ For second half-journey, $\text{WD} = \int_w^0 \dfrac{\left(-ma + Av^2 + R\right)v}{-a}\,\mathrm{d}v$ $= -mad + Rd + Aad^2$ Summing gives $2dR + 2Aad^2$ $R > ma \Rightarrow F = Av^2 + R - ma > 0$ always	**M1** **A1** **B1B1** **A1** **AG** **E1** **[6]**	Performing integration Correct answer in terms of d. B1 for correct limits B1 for correct integrand N.b. integrals may be combined to get to the same result.

(ii) If $R < ma$ then F is zero when $Av^2 = ma - R$ i.e. when $v = V = \sqrt{\dfrac{ma-R}{A}}$	**B1**	Finding an expression for the critical speed.
For F to fall to zero during motion, $V < w$ i.e. when $\dfrac{ma-R}{A} < 2ad$ i.e. $R > ma - 2Aad$	**E1** **E1**	
In this case, WD = $mad + Rd + Aad^2$, as before, for the first half-journey	**B1**	
For the second half WD = $\displaystyle\int_w^V \frac{\left(-ma + Av^2 + R\right)v}{-a}\,dv$	**M2**	
$\left[(ma-R)\dfrac{v^2}{2a} - \dfrac{Av^4}{4a}\right]_w^V$	**A1**	
$= \dfrac{1}{2a}(ma-R)\left(\dfrac{ma-R}{A}\right) - \dfrac{A}{4a}\left(\dfrac{ma-R}{A}\right)^2 -$ $\dfrac{1}{2a}(ma-R)(2ad) + \dfrac{A}{4a}\left(4a^2d^2\right)$ $= \dfrac{1}{2Aa}(ma-R)^2 - \dfrac{1}{4Aa}(ma-R)^2 - (ma-R)d +$ Aad^2	**M1**	Substituting expressions for V and w.
$= \dfrac{1}{4Aa}(ma-R)^2 - mad + Rd + Aad^2$	**A1 CAO**	Without wrong working
So total WD = $\dfrac{1}{4Aa}(ma-R)^2 + 2Rd + 2Aad^2$	**AG** **[9]**	

Question 11

		
(i)		
At A, $KE = \frac{1}{2}mu^2 = \frac{5}{2}mag, PE = 0$	**B1**	
At $A_1, K = \frac{1}{2}mv^2, PE = 2mag$	**B1**	
Conservation of energy: $$\frac{5}{2}mag = \frac{1}{2}mv^2 + 2mag$$	**M1**	
$$v^2 = ga$$		
$$v = \sqrt{ga}$$	**A1**	
	[4]	
If angle at A_1 is β and it just passes the second wall then we have:		
$$0 = v\sin\theta\, t - \frac{1}{2}gt^2$$	**M1**	Using $s = ut + \frac{1}{2}at^2$
So $t = \frac{2v}{g}\sin\beta$	**A1**	Solving for t at second wall.
Also, $a = v\cos\beta\, t$	**M1**	Considering horizontal distance
$$= \frac{2v^2\sin\beta\cos\beta}{g}$$		N.b. Some candidates may just quote this (or equivalent). Give full credit.
$$= 2a\sin\beta\cos\beta$$	**A1**	Combining previous results.
So $\sin(2\beta) = 1$	**A1**	
Therefore $\beta = 45°$	**AG**	Condone absence of domain considerations.
	[5]	
x velocity is constant so		
$$u\cos\alpha = vcos\,\beta$$	**M1**	Comparing x velocities
$$\sqrt{5ag}\cos\alpha = \sqrt{ag}\frac{1}{\sqrt{2}}$$ $$\cos\alpha = \frac{1}{\sqrt{10}}$$	**A1**	
$$\sin\alpha = \frac{3}{\sqrt{10}}, \tan\alpha = 3$$	**A1**	Converting to a more useful ratio.

Method 1: $$2a = \sqrt{5ag}\frac{3}{\sqrt{10}}t - \frac{1}{2}gt^2$$ $$= \frac{3\sqrt{ag}}{\sqrt{2}}t - \frac{1}{2}gt^2$$ So $$t^2 - \frac{3\sqrt{2a}}{\sqrt{g}}t + \frac{4a}{g} = 0$$	**M1**	Using $s = ut + \frac{1}{2}at^2$
$$\left(t - \sqrt{\frac{2a}{g}}\right)\left(t - 2\sqrt{\frac{2a}{g}}\right) = 0$$		
First time over the wall means that $t = \sqrt{\frac{2a}{g}}$	**A1**	
So $d = ucos\ \theta\ t = \sqrt{5ag} \times \frac{1}{\sqrt{10}} \times \sqrt{\frac{2a}{g}} = a$	**A1**	
Method 2: $$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$	**M1**	Using trajectory equation
$$2a = 3x - \frac{x^2}{a}$$	**A1**	Combining with previous results
$(x - a)(x - 2a) = 0$		
$x = a$	**A1**	
	[6]	
If the speed at h above first wall is v then by conserving energy, $$\frac{1}{2}5ag = \frac{1}{2}v^2 + (2a + h)g$$	**M1**	
$v^2 = ag - 2gh$	**B1**	
Using trajectory equation with origin at top of first wall and angle β as particle moves over first wall: $$y = h + x\ \tan\beta - \frac{gx^2(1 + \tan^2\beta)}{2v^2}$$ When $x = a$ we need $y = 0$: $$0 = h + a\ \tan\beta - \frac{ga^2(1 + \tan^2\beta)}{2v^2}$$	**M1**	Use of trajectory equation (might be several kinematics equations effectively leading to the same thing)
Treating this as a quadratic in $\tan\beta$: $$-\frac{ga^2}{2v^2}\tan^2\beta + a\tan\beta + h - \frac{ga^2}{2v^2} = 0$$ $$-ga^2\tan^2\beta + 2av^2\tan\beta + 2hv^2 - ga^2 = 0$$ The discriminant is: $$4a^2v^4 + 4ga^2(2hv^2 - ga^2)$$	**M1**	Considering the quadratic (or equivalently differentiating to find the max)
$$= 4a^2(g^2(a^2 - 4ah + 4h^2) + 2g^2h(a - 2h) - g^2a^2))$$ $$= 4a^2g^2(a^2 - 4ah + 4h^2 + 2ah - 4h^2 - a^2)$$ $$= -8a^3g^2h$$ $$< 0$$ Therefore no solution.	**A1**	Obtaining a clearly negative discriminant – this might take many alternative forms.
	[5]	

Question 12

(i) $$P(X+Y=n)=\sum_{r=0}^{n}P(X=r)(P(Y=n-r)$$	**B2**	
$$=\sum_{r=0}^{n}\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{n-r}}{(n-r)!}$$	**B1**	
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\frac{n!}{r!\,(n-r)!}\lambda^{r}\mu^{n-r}$$	**M1**	Attempting to manipulate factorials towards a binomial coefficient
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{r}\mu^{n-r}$$	**B1**	Identifying correct binomial coefficient
$$=\frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^{n}$$	**B1**	
Which is the the formula for $Po(\lambda+\mu)$	**E1**	Recognising result. Must state parameters
	[7]	
(ii)		
$$P(X=r\|X+Y=k)=\frac{P(X=r)\times P(Y=k-r)}{P(X+Y=k)}$$	**M2**	(may be implied by following line)
$$=\frac{\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}}{\frac{e^{-(\lambda+\mu)}}{k!}(\lambda+\mu)^{k}}$$	**A1**	
$$=\frac{k!}{r!\,(k-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^{r}\left(\frac{\mu}{\lambda+\mu}\right)^{k-r}$$	**A1**	
Which is a $B\left(k,\frac{\lambda}{\lambda+\mu}\right)$ distribution.	**E1**	Parameters must be stated.
	[5]	
(iii) This corresponds to r=1, k=1 from (ii)	**M2**	Can be implied by correct answer.
So probability is $\frac{\lambda}{\lambda+\mu}$.	**A1**	
(iv)	**[3]**	
Expected waiting time given that Adam is first is waiting time for first fish plus waiting time for Eve $\left(=\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)$	**B2**	Also accept waiting time given Eve is first. Must be clearly identified.
Expected waiting time is: E(Waiting time|Adam first)P(Adam first)+E(Waiting time|Eve first)P(Eve first)	**M2**	
$=\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)\times\frac{\lambda}{\lambda+\mu}+\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\times\frac{\mu}{\lambda+\mu}$	**A1**	
$$=\frac{1}{\lambda}+\frac{1}{\mu}-\frac{1}{\lambda+\mu}$$		No need for this algebraic simplification.
	[5]	

Question 13

(i)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{k-1}$	**M1A1**	M1 for any attempt relating to the geometric distribution – e.g. missing first factor or power slightly wrong.
$= pq^{k-1}$ Where $p = \frac{1}{n}, q = 1 - \frac{1}{n}$		Although not strictly necessary, you may see this substitution frequently
Expected number of attempts is given by $p + 2pq + 3pq^2 \dots$	**M1**	May be written in sigma notation
$= p(1 + 2q + 3q^2 \dots)$		
$= p(1-q)^{-2}$	**M1**	Linking to binomial expansion
$= \frac{p}{p^2} = \frac{1}{p}$		
$= n$	**A1**	
	[5]	
(ii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}$ for $k = 1 \dots n$	**B1**	
Expected number of attempts is given by $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} \dots + \frac{n}{n}$	**M1**	
$= \frac{n+1}{2}$	**M1A1**	M1 for clearly recognising sum of integers / arithmetic series.
	[4]	
(iii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt})$ $= \frac{n-1}{n} \times \frac{n}{n+1} \times \frac{n+1}{n+2} \dots \times \frac{1}{n+k-1}$	**M1** **A1**	M1 for an attempt at this, possibly by pattern spotting the first few cases. Condone absence of checking $k = 1$ case explicitly.
$= \frac{n-1}{(n+k-2)(n+k-1)}$	**M1** **AG**	M1 for attempting telescoping (may be written as an induction)
$= (n-1)\left(\frac{-1}{n+k-1} + \frac{1}{n+k-2}\right)$	**M2** **A1**	Attempting partial fractions (This may be seen later)
	[6]	
Expected number of attempts is given by $(n-1)\sum_{1}^{\infty}\left(\frac{k}{n+k-2} - \frac{k}{n+k-1}\right)$	**M1**	
$= (n-1)\left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) \dots\right]$		
$= (n-1)\left[\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} \dots\right]$	**M1A1**	M1 for attempting telescoping
$= (n-1)\left(\sum_{r=1}^{\infty}\frac{1}{r} - \sum_{r=1}^{n-2}\frac{1}{r}\right)$	**B1**	
In the brackets there is an infinite sum minus a finite sum, so the result is infinite.	**E1**	
	[5]	